Oscillations in an $x^{(2n+2)/(2n+1)}$ Potential via He's Frequency-Amplitude Formulation

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A recent technique, known as He's frequency-amplitude formulation approach, is proposed in this letter to obtain an analytical approximate periodic solution to a nonlinear oscillator equation with potential of arbitrary fractional order. The solution procedure of the present approach is very simple and more convenient in comparison with the harmonic balance method.

Key words: Nonlinear Oscillators; He's Frequency-Amplitude Formulation; Periodic Solution.

1. Introduction

In this letter, we consider nonlinear oscillators that can be modelled by equations of motion, where the potential takes the form

$$V(x) = V_0 x^{(2n+2)/(2n+1)}, \quad n = 1,2,3,\ldots, \tag{1}$$

where V_0 is a positive constant. In this case, the force derived from (1) is given as [1]

$$f(x) = -\left(\frac{2n+2}{2n+1}\right) V_0 x^{1/(2n+1)}. \tag{2}$$

By this, a particle of mass M acted on by the force of (2), has the equation of motion

$$M\frac{\mathrm{d}^2x}{\mathrm{d}x^2} + \left(\frac{2n+2}{2n+1}\right) V_0 x^{1/(2n+1)} = 0. \tag{3}$$

This equation can be transformed to the dimensionless form [1]

$$\frac{\mathrm{d}^2\bar{x}}{\mathrm{d}\bar{x}^2} + \bar{x}^{1/(2n+1)} = 0. \tag{4}$$

Dropping the bars, then we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}x^2} + x^{1/(2n+1)} = 0. \tag{5}$$

In [1], Mickens applied the method of harmonic balance to obtain an approximate analytical periodic solution for (5) under the initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{6}$$

In this letter, we aim to show how to apply the frequency-amplitude formulation developed recently by He [2, 3] and used later by many authors [4 – 9] to investigate several nonlinear oscillators, to solve (5) and (6). Before launching into the main idea of the present letter, we rewrite (5) in the form

$$\left(\frac{\mathrm{d}^2x}{\mathrm{d}x^2}\right)^{(2n+1)} + x = 0. \tag{7}$$

According to the frequency-amplitude formulation approach, we use the following trial function to determine the frequency-amplitude, $(\omega - A)$, relationship

$$x = A\cos(\omega t). \tag{8}$$

Substituting (8) into (7), we obtain the following residual:

$$R(t) = -\left[A\omega^2\cos(\omega t)\right]^{(2n+1)} + A\cos(\omega t). \tag{9}$$

We then set [7 – 9]

$$\int_0^{T/4} R(t)\cos(\omega t)\mathrm{d}t = 0,\ T = 2\pi/\omega. \tag{10}$$

Using (9) into (10), yields

$$\begin{aligned} &-(A\omega^2)^{(2n+1)}\int_0^{\pi/2\omega}(\cos(\omega t))^{(2n+2)}\,\mathrm{d}t \\ &+A\textstyle\int_0^{\pi/2\omega}\cos^2(\omega t)\mathrm{d}t = 0. \end{aligned} \tag{11}$$

Therefore

$$-(A\omega^2)^{(2n+1)}\frac{\sqrt{\pi}\,\Gamma(n+3/2)}{2\Gamma(n+2)} + \frac{\pi A}{4} = 0. \tag{12}$$

Hence

$$\omega = \left[\frac{\sqrt{\pi}\,\Gamma(n+2)}{2A^{2n}\Gamma(n+3/2)}\right]^{1/(4n+2)}, \tag{13}$$

which is the same frequency-amplitude, $(\omega - A)$, relationship obtained by the harmonic balance method [1], but in more easier way.

2. Conclusion

In this note, we have applied a new approach, named He's frequency-amplitude formulation, to obtain an approximate periodic solution to a nonlinear oscillator equation with potential of fractional order. We also showed in this letter that the analytical approximation is obtained easily and elegantly by this new method if compared with the harmonic balance method.

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